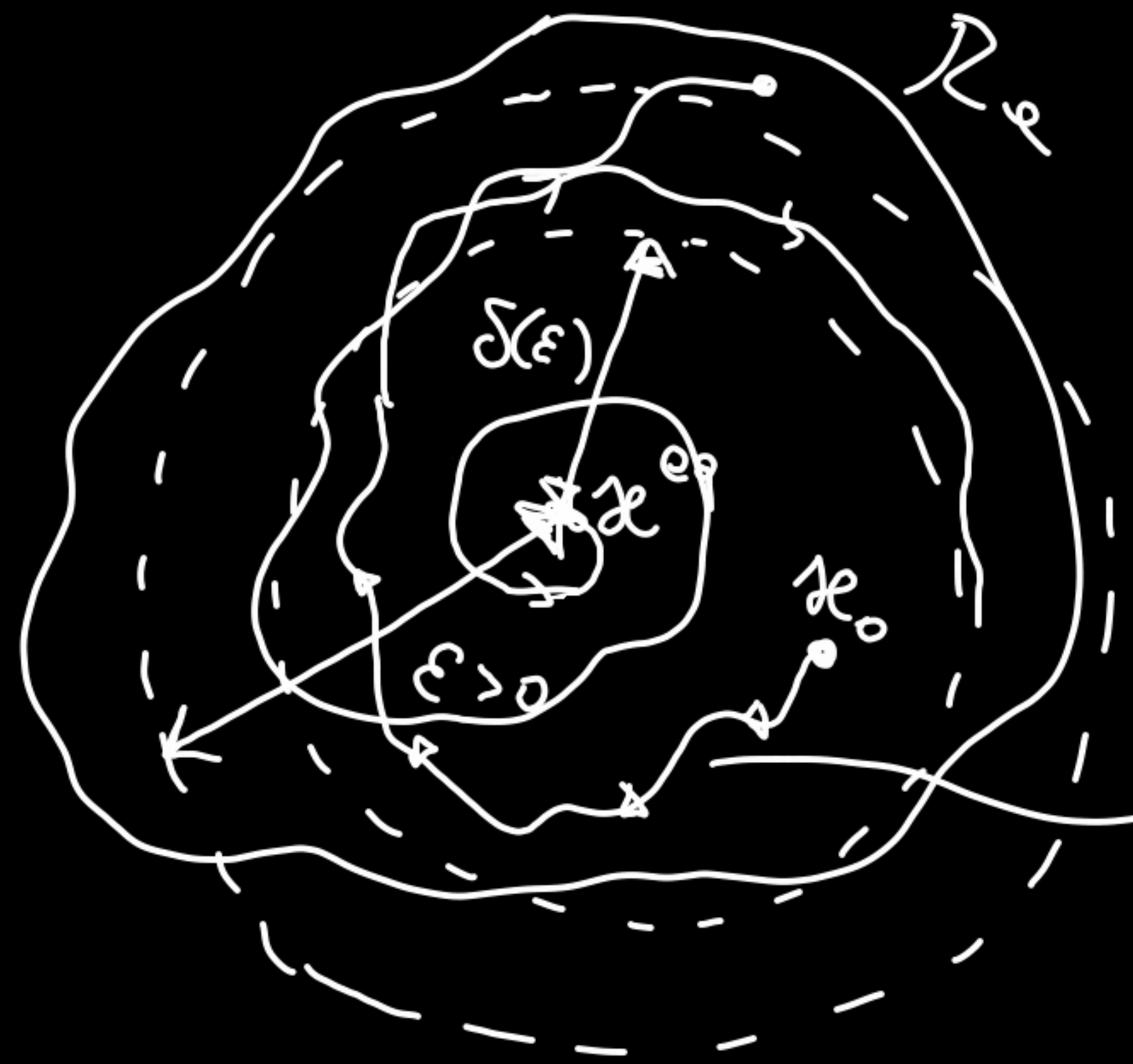


12/10/2026 - STABILITY OF MPC

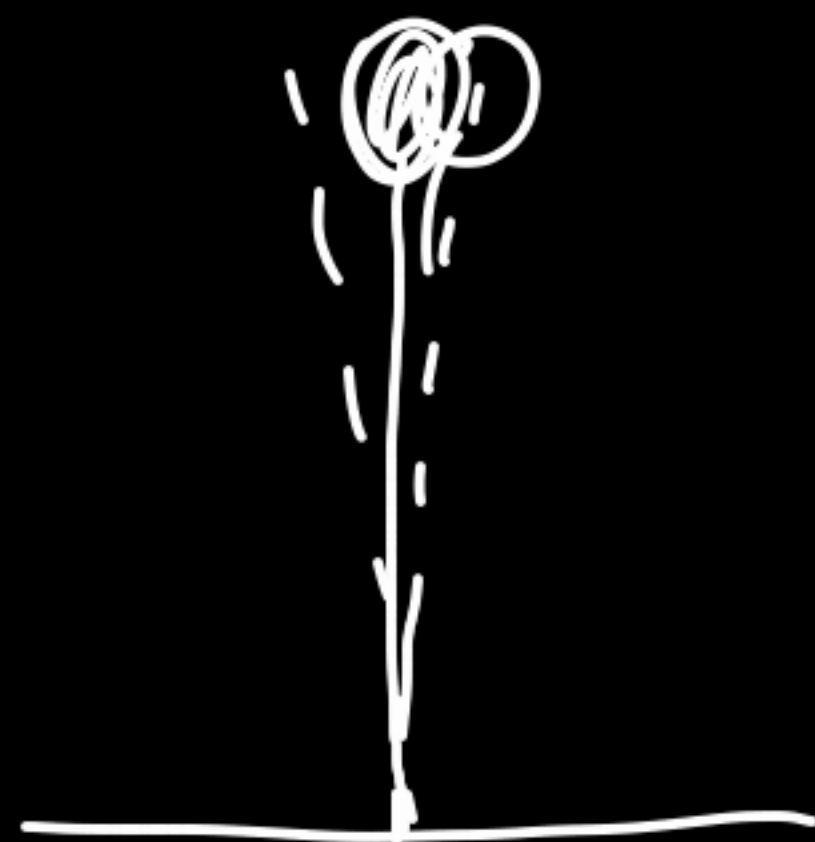
DEF. STABILITY ($x^{eq} = 0$)



$$x_{k+1} = f(x_k)$$



(5)



IF x^{eq} IS UNIQUE, AS,
AND $\mathcal{R}^2 \equiv X \rightarrow GAS$
(GLOBALLY ASYMPTOTICALLY
STABLE)

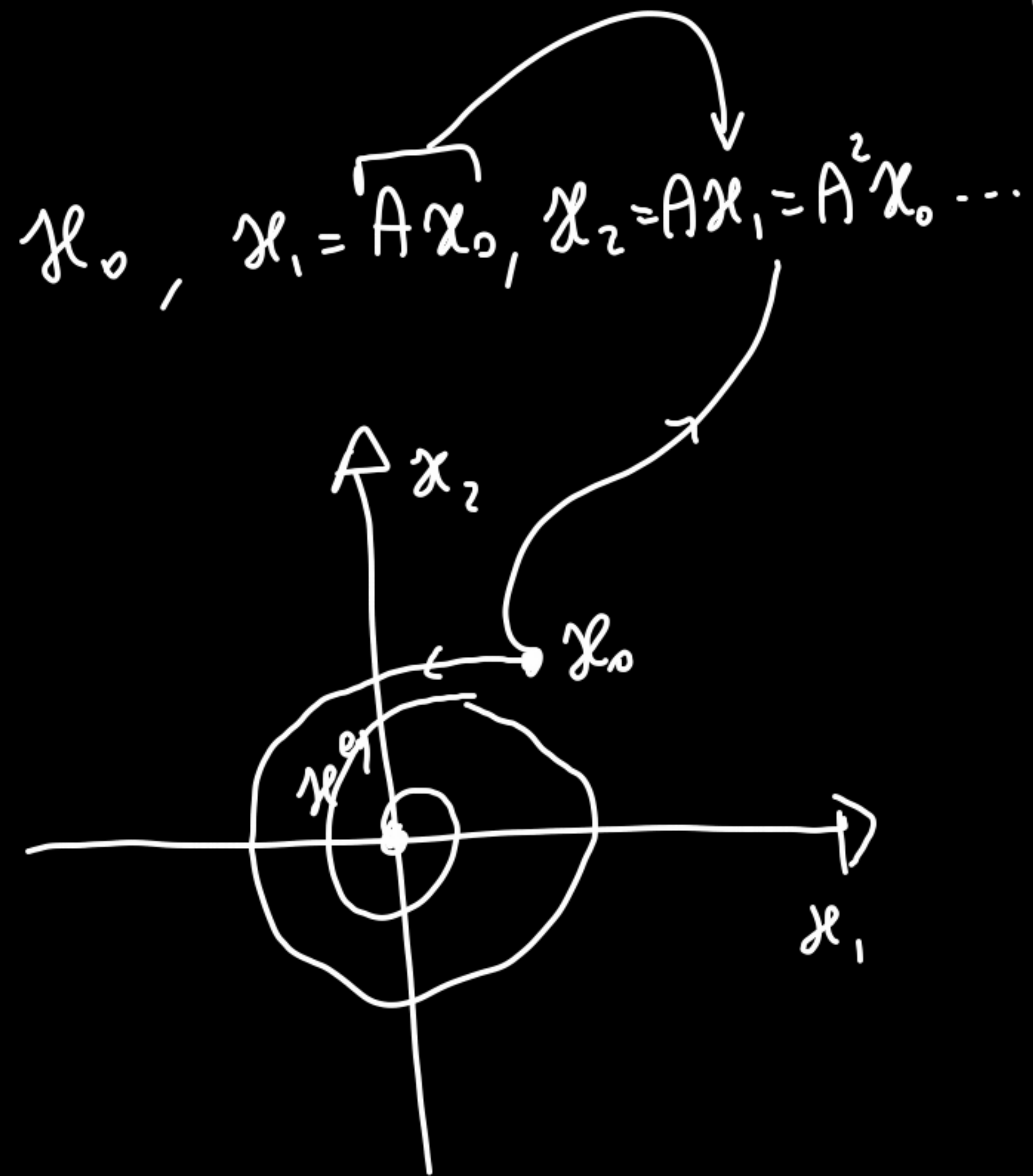
$$\forall \epsilon > 0 \exists \delta(\epsilon) > 0 : \|x_{k_0} - x^{eq}\| \leq \delta(\epsilon) \rightarrow \|x_k - x^{eq}\| \leq \epsilon, \forall k \geq k_0$$

STABILITY CRITERIA:

x^{eq} EQ. POINT OF $x_{k+1} = f(x_k)$

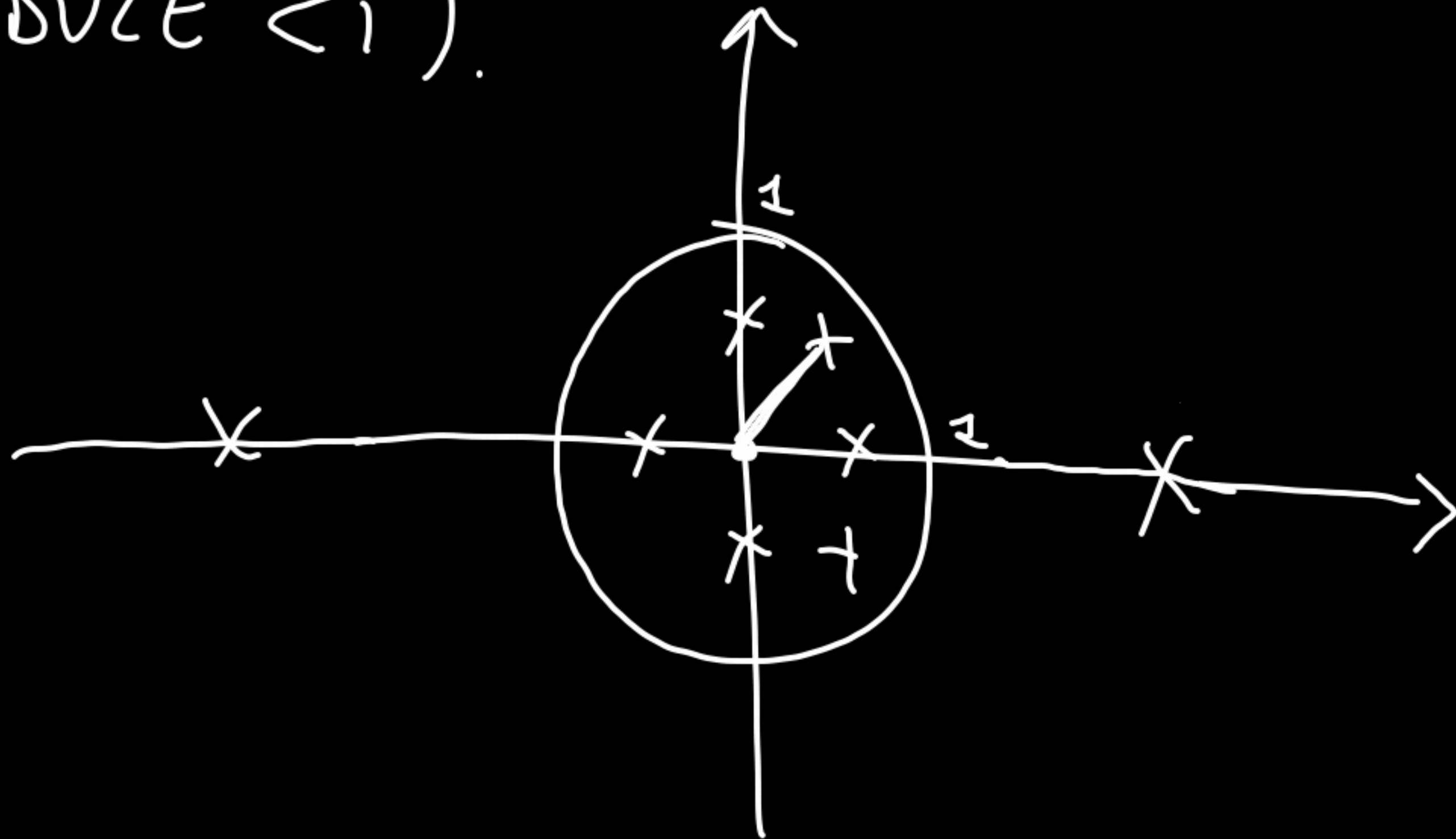
• LINEAR SYSTEM $x_{k+1} = A x_k$
($x^{eq} = 0$) SOL. $x_k = A^k x_0, \forall k$

STABILITY DEPENDS ON THE
EIGENVALUES OF A ("AUTOVALORI")



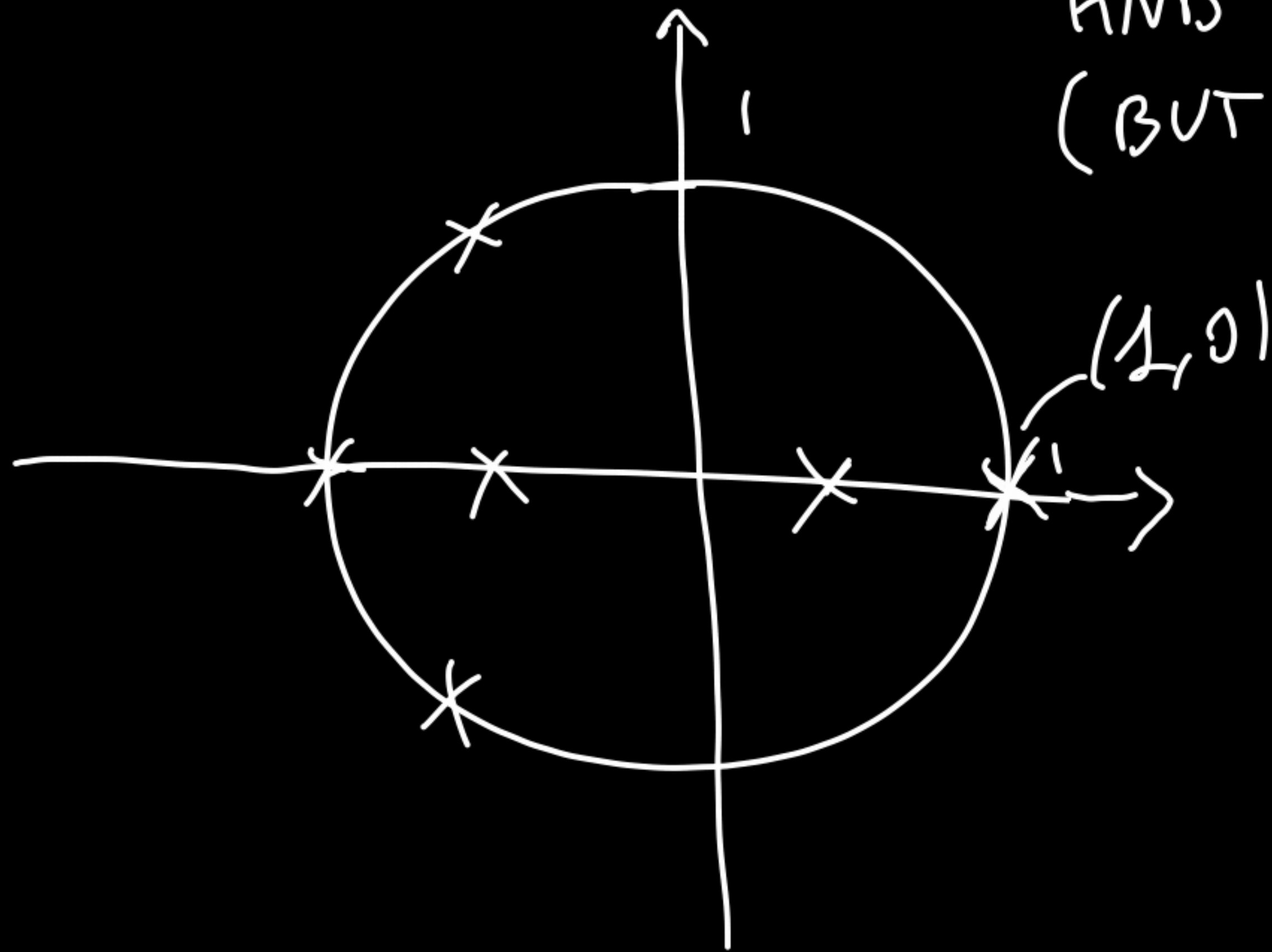
STAB. CRITERION LINEAR SYSTEM ($x_{k+1} = Ax_k$)

- $(x^e = 0)$ is AS IFF ALL THE EIGENVALUES OF A
ARE INSIDE THE UNITARY CIRCLE (ALL EIGS. HAVE
MODULE < 1).

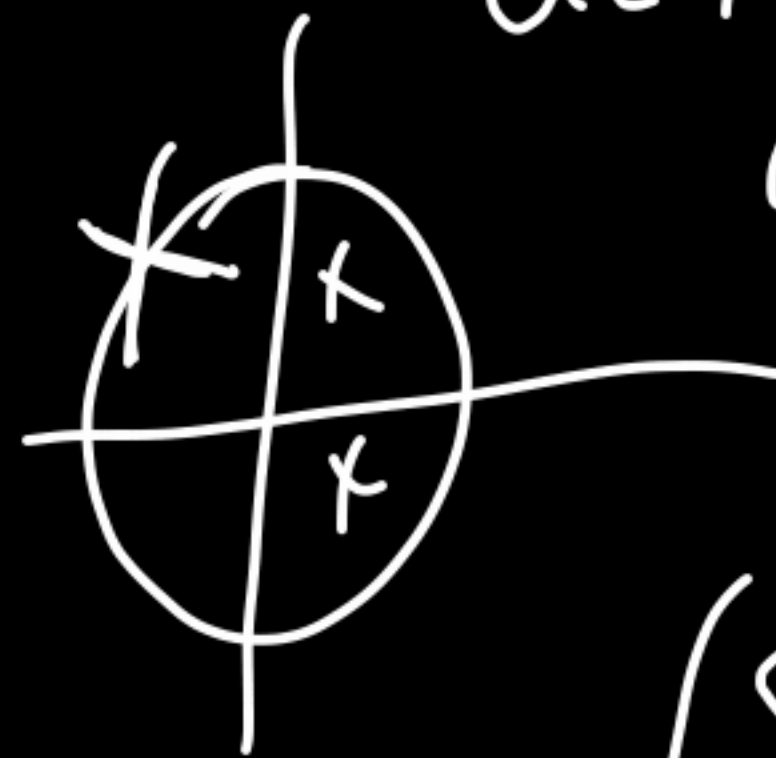


• UNSTABLE : $\Leftarrow$ IF AT LEAST ONE EIG. IS OUTSIDE OF THE UNITARY CIRCLE.

• S (SIMPLE STABILITY) : • ALL EIGS ARE INSIDE THE CIRCLE AND ONE OR MORE ON THE BORDER (BUT WITH MULTIPLICITY ONE)



• ALL EIGS INSIDE, PLUS ONE OR MORE COINCIDENT ONES ON BORDER



$\Downarrow$ OTHER
(S, U) CHECKS NEEDED

WE WORK WITH NONLINEAR SYSTEMS.

TWO REASONS:

- $f(x_k, u_k)$ MIGHT BE NONLINEAR

- MPC IS A NONLINEAR CONTROLLER.

WE NEED A
STABILITY CRITERION
FOR NONLINEAR
SYSTEMS

$u_{0|k}^*$ IS THE SOLUTION OF $P_N(x_k) \rightarrow u_{0|k}^* = K_N^*(x_k)$

h_N IN GENERAL IS A NONLINEAR FUNCTION.

$$x_{k+1} = \overset{L}{f}\left(x_k, \overset{NL}{h_N}(x_k)\right) = \overset{NL}{\tilde{f}}(x_k) \rightarrow \text{MODEL OF SYSTEM CONTROLLED WITH MPC.}$$

LYPUNOV CRITERION OF STABILITY FOR NONLINEAR SYSTEMS. ($x^e = 0$)

DEFINITIONS. LET'S CONSIDER A SCALAR FUNCTION OF THE STATE : $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}$.

(DEF.) POSITIVE DEFINITE FUNCTION (GLOBAL) ($V > 0$)

A FUNCTION $V : \mathbb{R}^n \rightarrow \mathbb{R}$ IS POSITIVE DEFINITE ($V > 0$) IFF:

- ① $V(0) = 0$;
- ② $V(x) > 0 \quad \forall x \neq 0$.

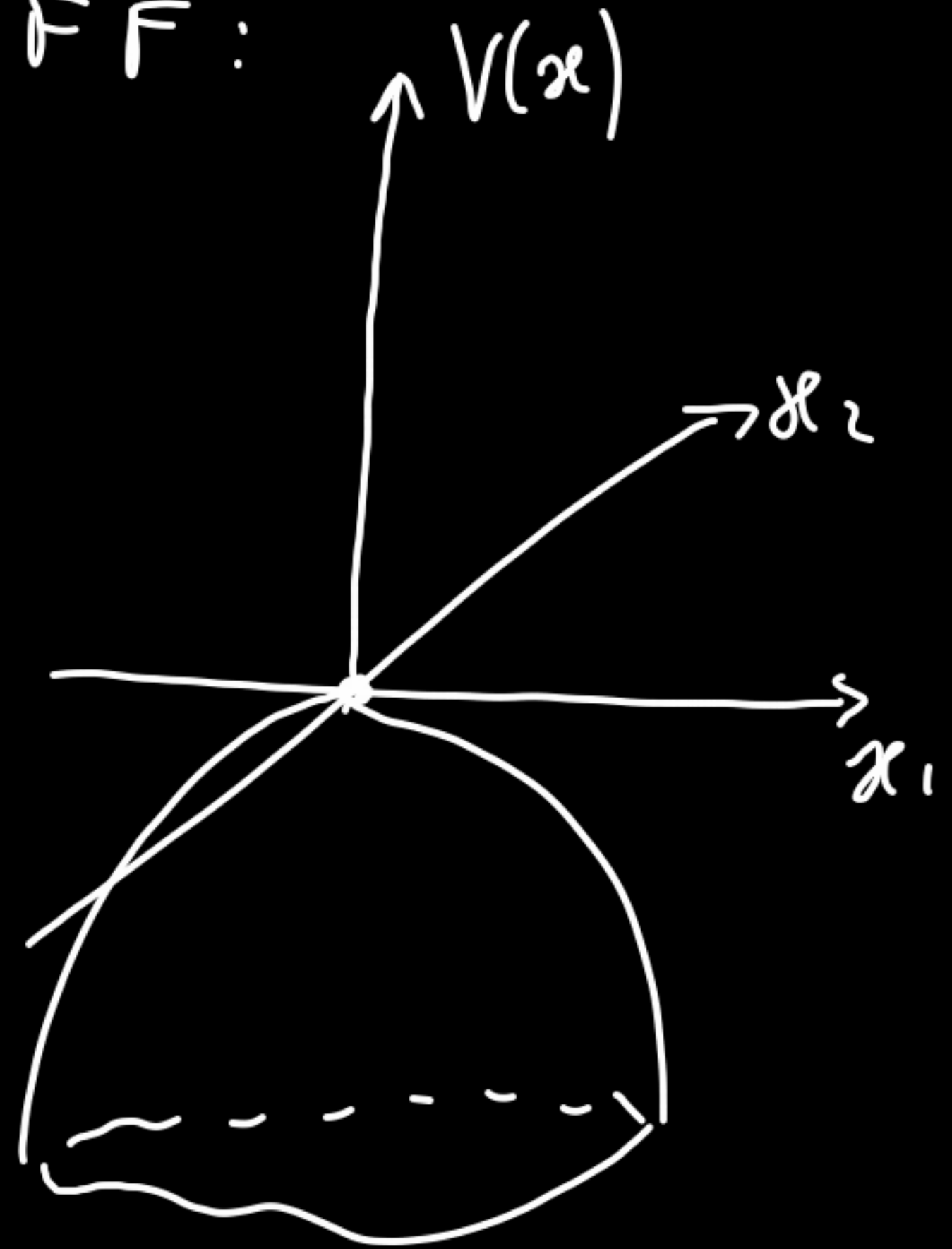


DEF. NEGATIVE-DEFINITE FUNCTIONS ($V < 0$)

V IS NEGATIVE DEFINITE ($V < 0$) IFF:

① $V(0) = 0$;

② $V(x) < 0$, $\forall x \neq 0$.



DEF. SEMI-POSITIVE DEFINITE FUNCTIONS ($V \geq 0$)

V IS SEMI-POSITIVE DEFINITE IFF:

① $V(0) = 0$;

② $V(x) \geq 0, \forall x$.

DEF. SEMI-NEGATIVE DEFINITE FUNCTION ($V \leq 0$)

V IS SEMI-NEGATIVE DEF. IFF:

① $V(0) = 0$;

② $V(x) \leq 0, \forall x$.

$(-V \geq 0)$

PARTICULAR CASES FOR V .

EXAMPLE:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$V(x) = \underbrace{2x_1^2 + 3x_2^2}_{\text{SCALAR}} = (x_1 \ x_2) \overbrace{\begin{pmatrix} \textcircled{2} & 0 \\ 0 & \textcircled{3} \end{pmatrix}}^P \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =$$

$$= x^T P x \quad \text{QUADRATIC FORM.}$$

$$V(x) = x^T P x > 0 \iff \text{ALL EIGENVALUES OF } P \text{ ARE POSITIVE.}$$

EXAMPLE

$$V(x) = 2 \cdot (x_1)^2 + 3x_2^2 \quad ??$$

① $V(0) = ?$ ✓

② $V(x) \geq 0 \quad \forall x \neq 0.$

$$\begin{pmatrix} \pi \\ 0 \end{pmatrix} \neq 0 \rightarrow V\left(\begin{pmatrix} \pi \\ 0 \end{pmatrix}\right) = 0$$

WE GAVE GLOBAL DEFINITIONS

DEF. A LOCALLY POSITIVE-DEFINITE FUNCTION.

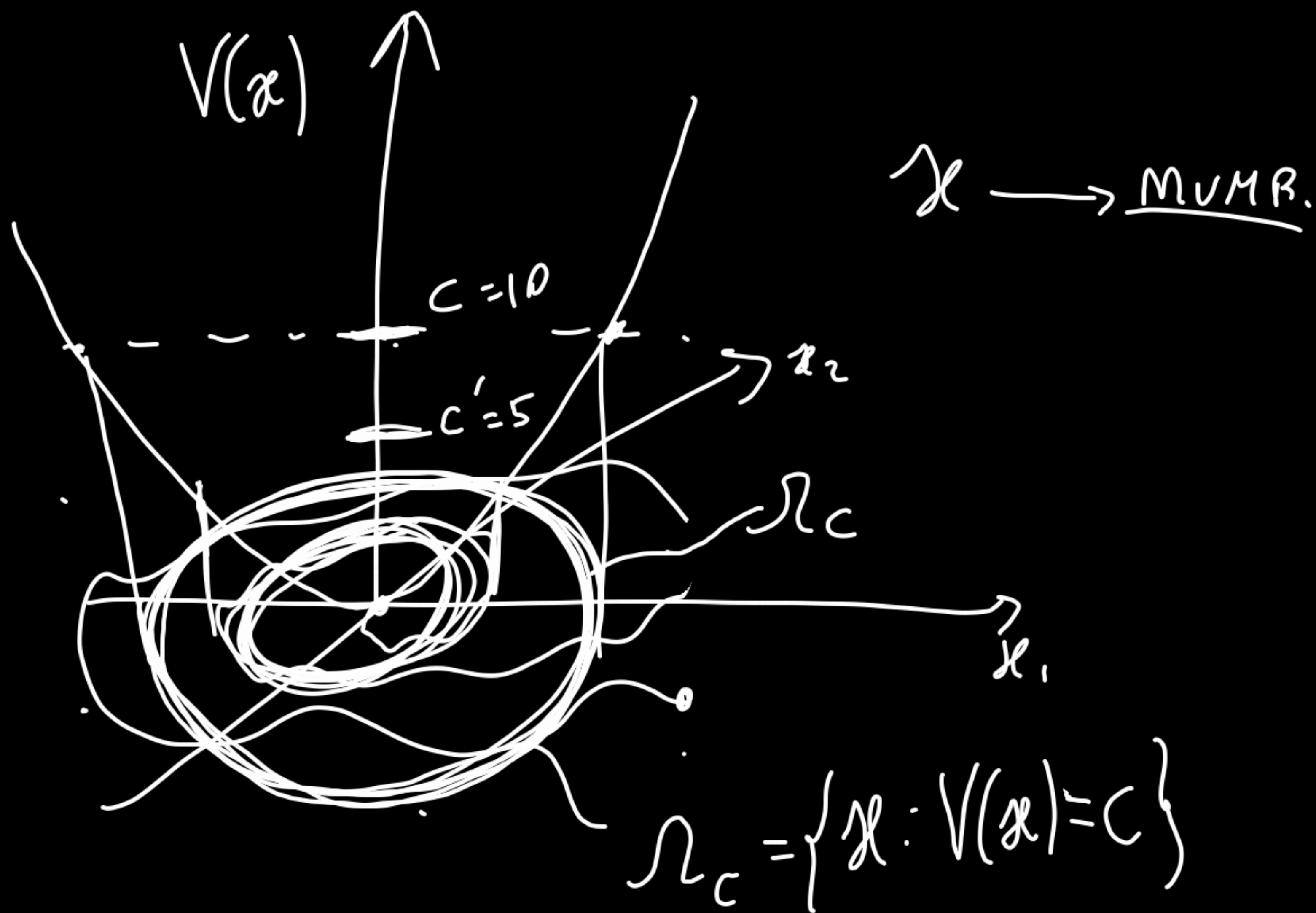
$V: \mathbb{R}^n \rightarrow \mathbb{R}$ IS LOCALLY POSITIVE-DEFINITE AROUND x^{eq} IFF.

① $V(x^{eq}) = 0$;

② $V(x) > 0$, $\forall x$ IN A REGION AROUND x^{eq} .



$\Omega_{c'}$



LEVEL SET

LYAPUNOV THEOREM FOR SIMPLE STABILITY OF x^{eq}

GIVEN $x_{k+1} = f(x_k)$ NONLINEAR, AN EQUILIBRIUM POINT x^{eq} OF THE SYSTEM IS STABLE IF ^{SUFFICIENT CONDITION} WE CAN FIND:

- ① A FUNCTION $V(x): \mathbb{R}^n \rightarrow \mathbb{R}$ THAT IS (AT LEAST) LOCALLY POSITIVE DEFINITE AROUND x^{eq} ($V(x^{eq}) = 0$, $V(x) > 0$, $\forall x$ "NEAR" x^{eq}).
- ② THE INCREMENT OF V , $\Delta V(x) \triangleq V(f(x)) - V(x)$ IS A SEMI-NEGATIVE FUNCTION (AT LEAST LOCALLY) AROUND x^{eq} .



LYAPUNOV THEOREM FOR AS OF x^{eq} .

AN EQ. POINT OF A SYSTEM $x_{k+1} = f(x_k)$ IS AS IF WE CAN FIND A FUNCTION $V: \mathbb{R}^n \rightarrow \mathbb{R}$:

① //

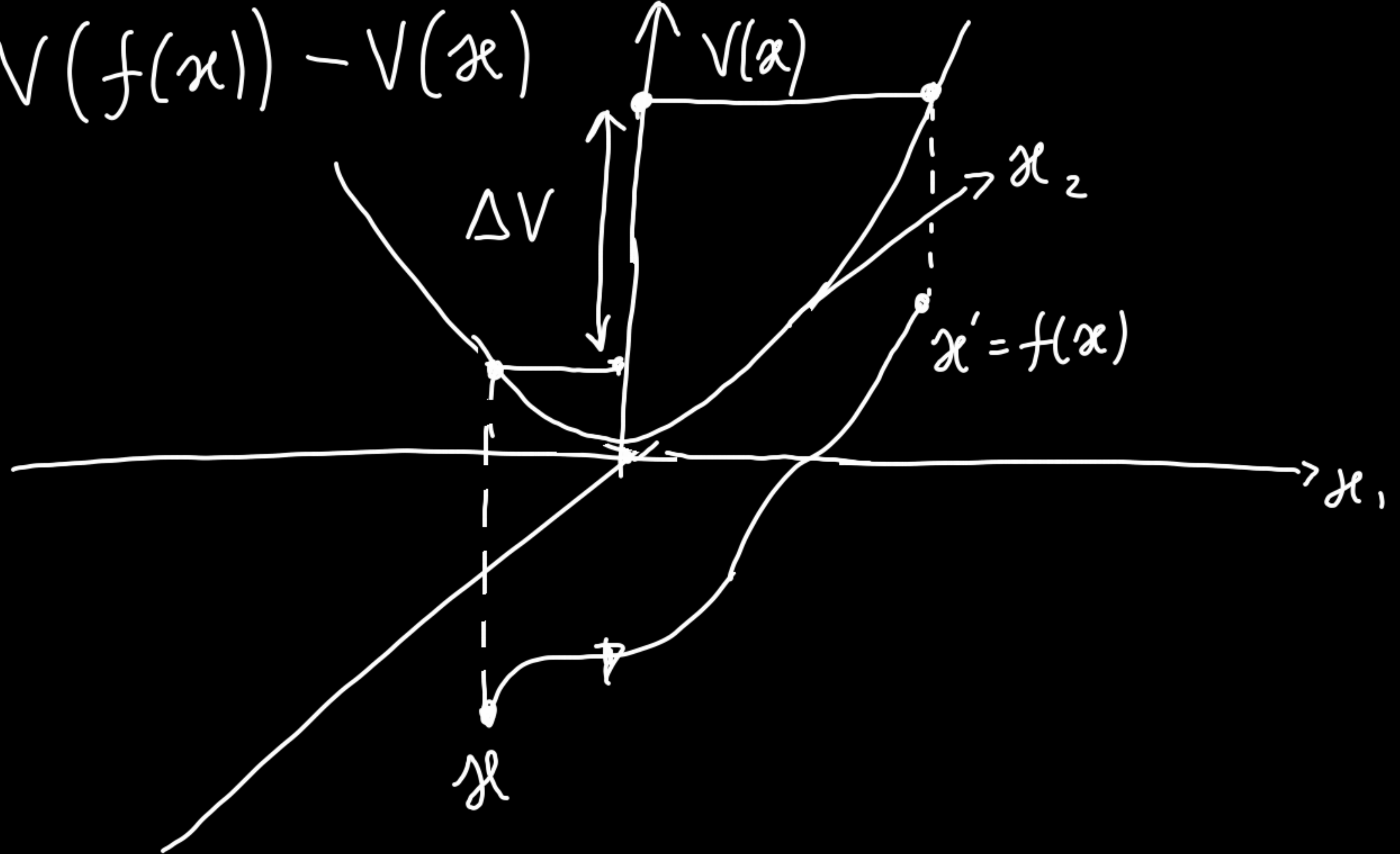
② $\Delta V(x) < 0$ (AT LEAST) LOCALLY AROUND x^{eq} .

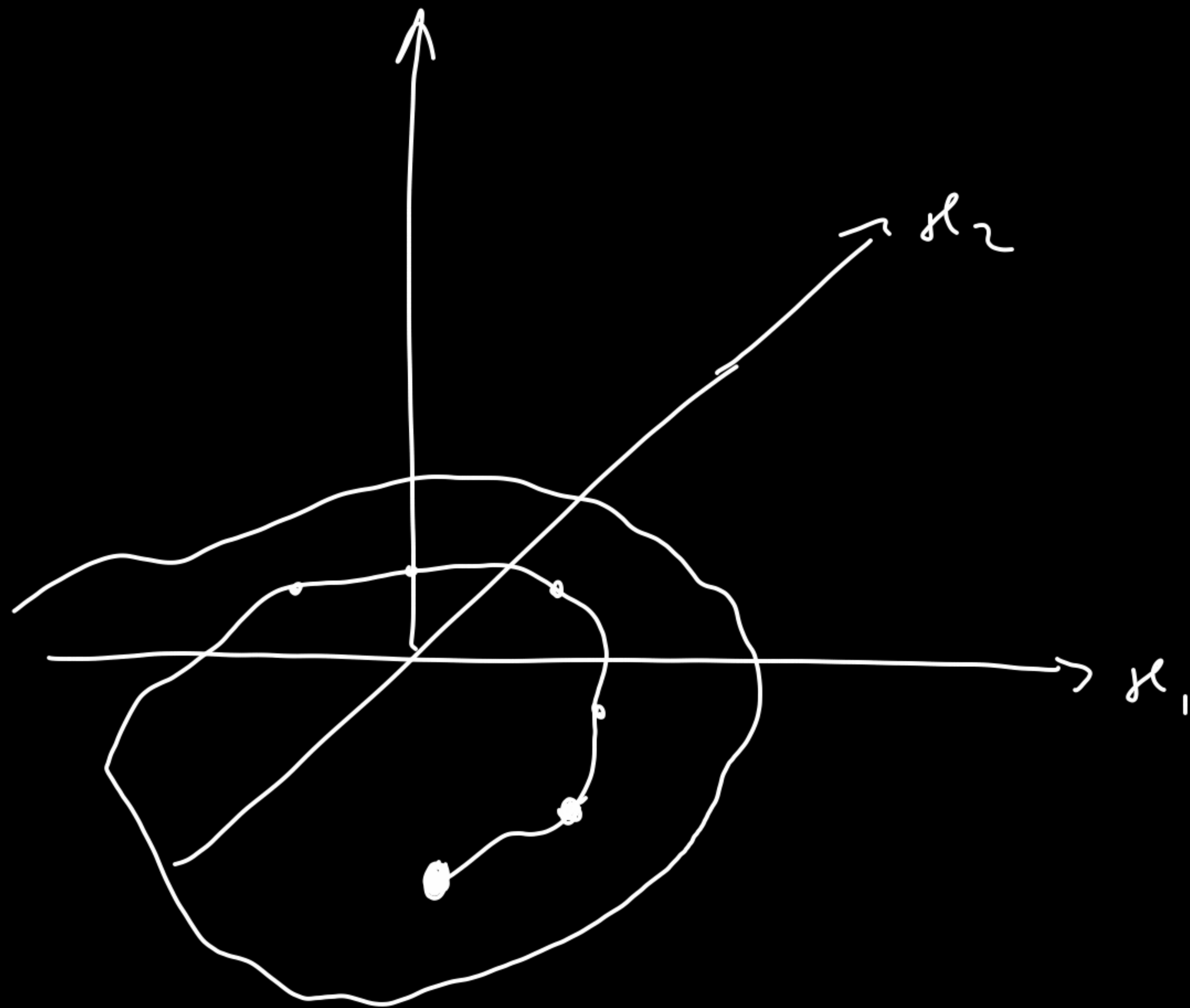
WE HAVE TO SEARCH FOR A FUNCTION V THAT SATISFIES ① AND ②. ANY FUNCTION WE TRY IS CALLED "CANDIDATE LYAPUNOV FUNCTION". FUNCTIONS THAT SATISFY 1 AND 2 ARE "LYAPUNOV FUNCTIONS".

$\Delta V(x)$ IS THE INCREMENT OF V COMPUTED ALONG THE
TRAJECTORIES OF THE SYSTEM $f(x)$. $x_{k+1} = f(x_k)$

$$x_{k+1} = f(x_k)$$

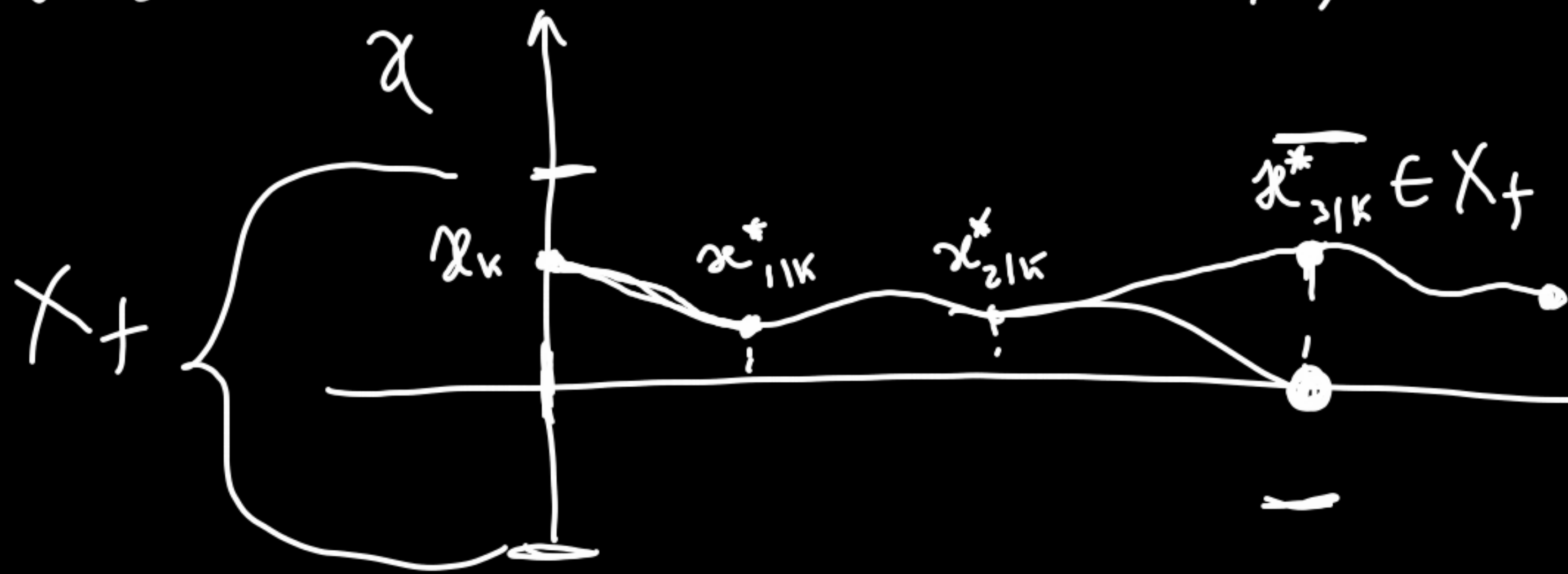
$$\underline{\Delta V(x)} = V(f(x)) - V(x)$$



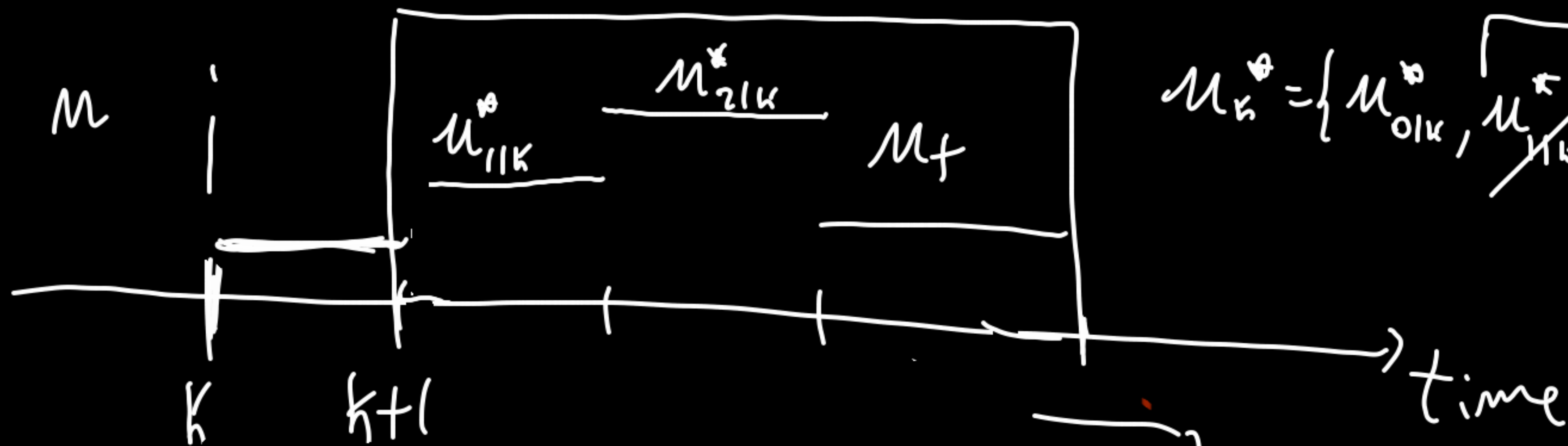


(QUESTION ON REC. FEASIBILITY)

$$M_f \in U$$



POSSO TROVARE UN U_f :
 $f(x_{Nk}^*, M_f) \in X_f, \forall$
 $x_{Nk}^* \in X_f$?

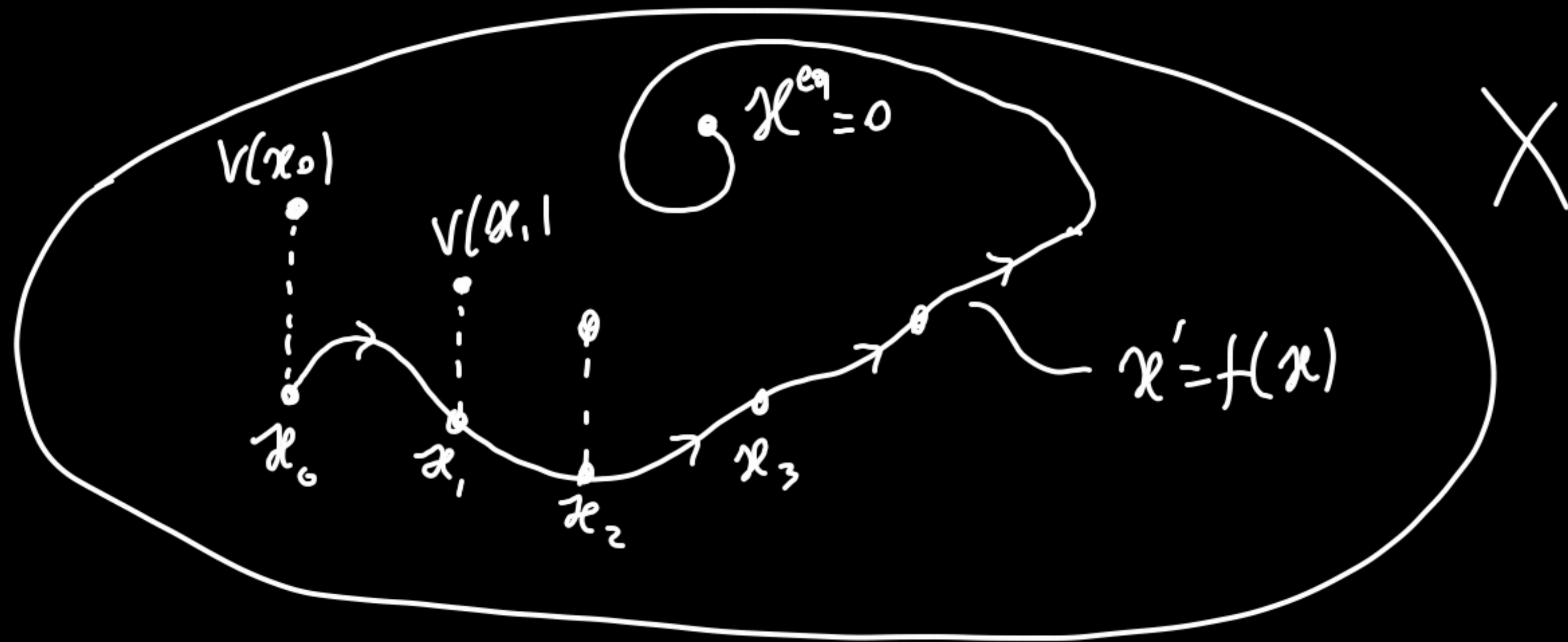


$$M_k^* = \{M_{0k}^*, \cancel{M_{1k}^*}, M_{2k}^*\}$$

$$\{I\} X$$

12/10/2026 - PART 2

$$\underline{\Delta V(x)} \triangleq V(f(x)) - V(x) \begin{cases} > 0 ? \\ < 0 ? \end{cases}$$



EXAMPLE:

$$\begin{cases} x_{1,k+1} = 2x_{1,k} \\ x_{2,k+1} = x_{2,k} \end{cases}$$

$$V(x) = x_1^2 + x_2^2 = x^T I x$$

TRY TO COMPUTE $\Delta V(x)$. IS IT > 0 ?

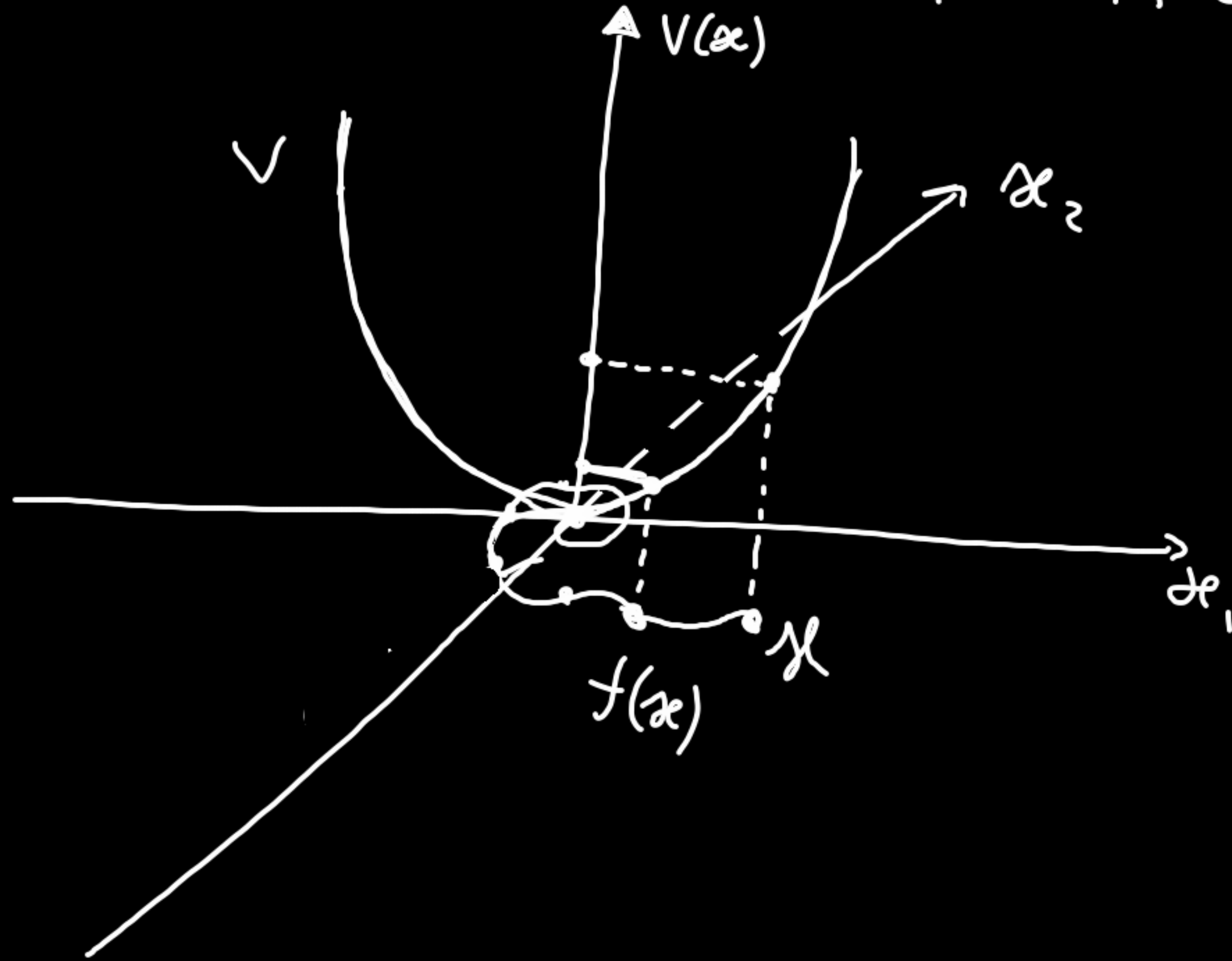
LYAPUNOV THEOREM FOR AS OF x^e .

x^{eq} IS A S EQUILIBRIUM OF $x_{k+1} = f(x_k)$ IF WE CAN FIND A FUNCTION $V: \mathbb{R}^n \rightarrow \mathbb{R}$:

① $V > 0$ AROUND x^{eq} (AT LEAST LOCALLY). SIMPLE

② $\Delta V(x) < 0$ AROUND x^{eq} (AT LEAST LOCALLY). DIFFICULT

INTUITIVE PROOF. LET'S SAY WE MANAGED TO FIND A V THAT SATISFIES ① (SIMPLE) AND ②. THEN, LET'S SEE x^{eq} IS ACTUALLY AS.



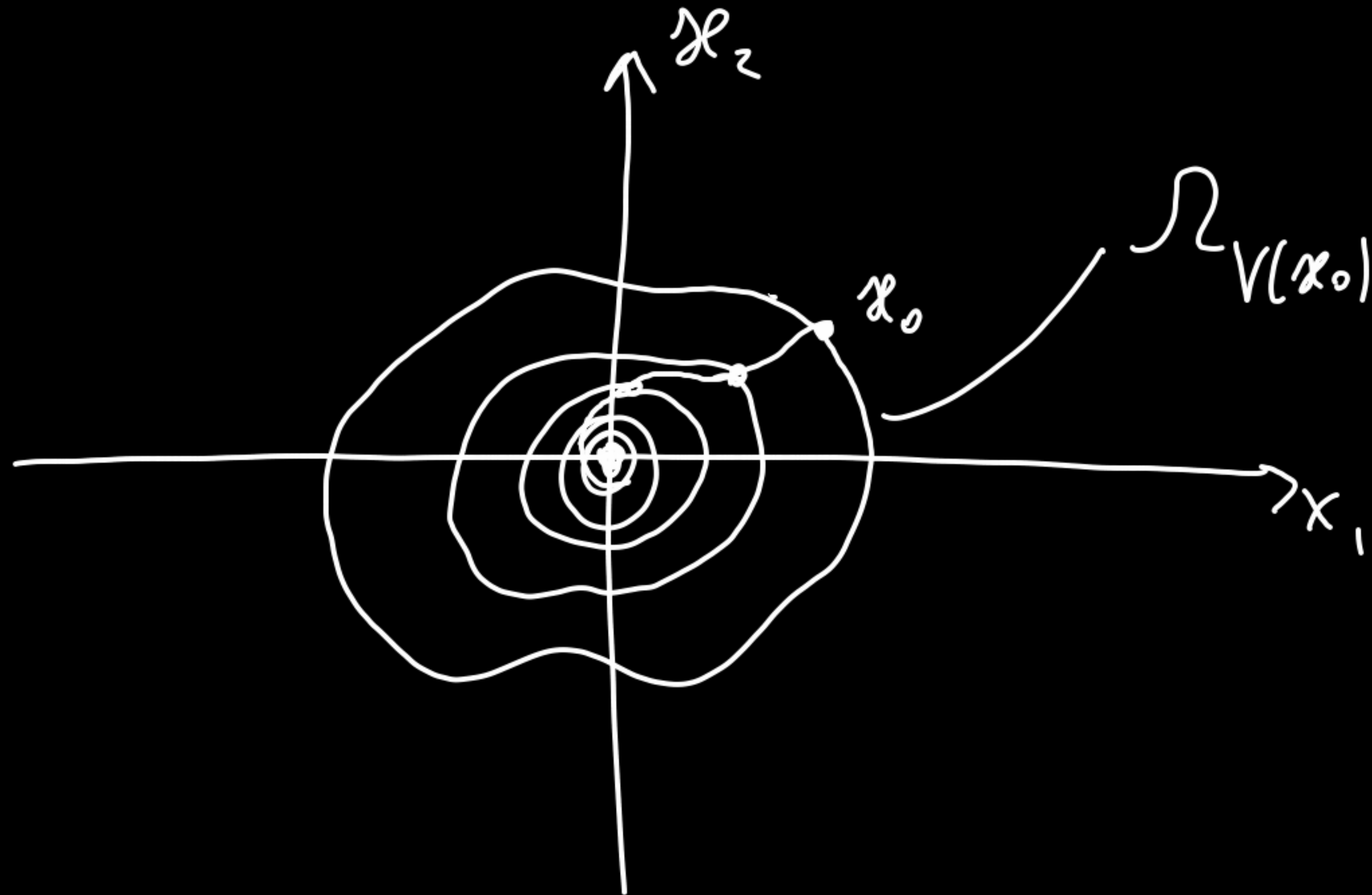
V IS STRICTLY DECREASING ALONG THE TRAJECTORIES OF THE SYSTEM

$$\hookrightarrow V \rightarrow 0$$

$$\downarrow x_k \rightarrow 0$$

$\rightarrow 0$ IS AS.

SINCE ② (V IS DECREASING)
 $V(x_1) < V(x_0)$



NOW WE APPLY LYAPUNOV THEOREM OF (SIMPLE) STABILITY TO MPC.

ASSUMPTIONS :

- CONSIDER SYSTEM $x_{k+1} = f(x_k, u_k)$ AND ASSUME IT HAS AN EQ. IN ZERO ($f(0,0) = 0$).
- WE WANT TO REGULATE STATE TO ZERO (CLASSIC TRACKING / REGULATION MPC).

REFERENCE MPC FORMULATION

$$P_N(x_k) : \min_{u_k = \{u_{0|k}, \dots, u_{N-1|k}\}} \left\{ \underbrace{\sum_{i=0}^{N-1} \ell(x_{i|k}, u_{i|k})}_{\substack{\cdot \\ V_N(x_k, \{u_{0|k}, u_{1|k}, \dots, u_{N-1|k}\})}} + V_f(x_{N|k}) \right\}$$

s.t.

$$\begin{cases} x_{i+1|k} = f(x_{i|k}, u_{i|k}), & i=0, 1, \dots, N-1 \\ x_{0|k} = x_k \end{cases}$$

$$\begin{cases} u_{i|k} \in U, & i=0, 1, \dots, N-1 \\ x_{i|k} \in X, & i=0, 1, \dots, N \end{cases}$$

$$\cdot x_{N|k} \in X_f$$

REGULATION MPC $\rightarrow$ WE WANT TO BRING x_k TO 0.

↓
2) ASSUMPTIONS ON ℓ, V_f :

- $\ell(x, u)$ IS SUCH THAT:

- $\ell(0, 0) = 0$

- $\ell(x, u) > 0, \quad \forall (x, u) \neq (0, 0)$

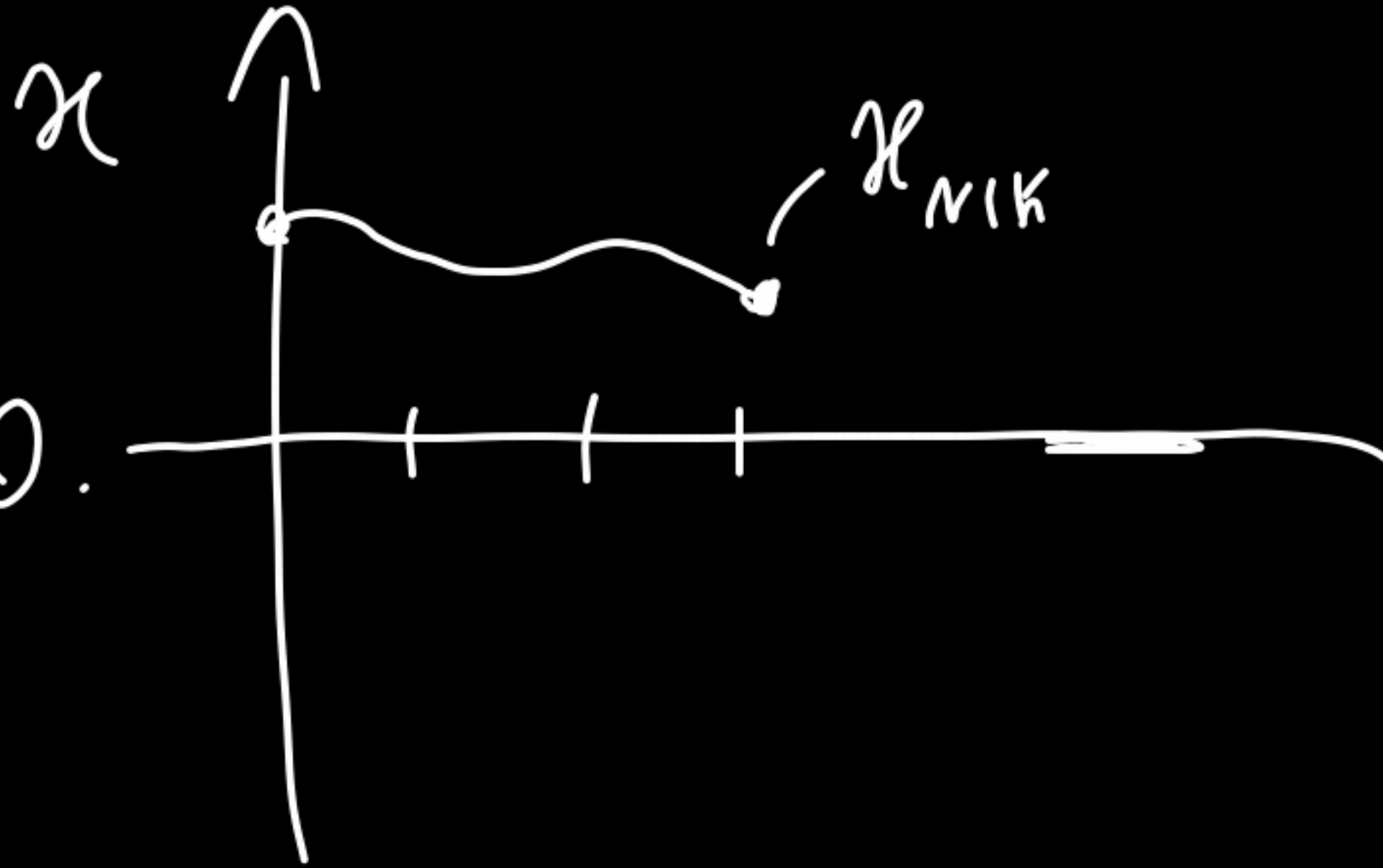
1) $\ell(x, u)$ IS POSITIVE DEFINITE FUNCTION

ASSUMPTIONS

2) $V_f(x)$ POSITIVE DEFINITE FUNCTION.

- $V_f(0) = 0$

- $V_f(x) > 0$, $\forall x \neq 0$.



RECAP OF ASSUMPTIONS:

- $x^{eq} = 0$ ($f(0,0) = 0$)

- $\rho(x,u)$ IS A POSITIVE DEF. OF x AND u , AROUND $(0,0)$
- $V_f(x)$ IS POSITIVE DEFINITE AROUND x^{eq} .

RESUR-
ACTION

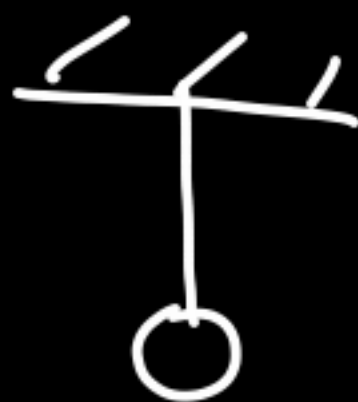
PROBLEM: CAN WE PROVE THAT MPC (SEE REFERENCE FORMULATION)
IS SUCH THAT x^{eq} IS STABLE?

FREE SYSTEM

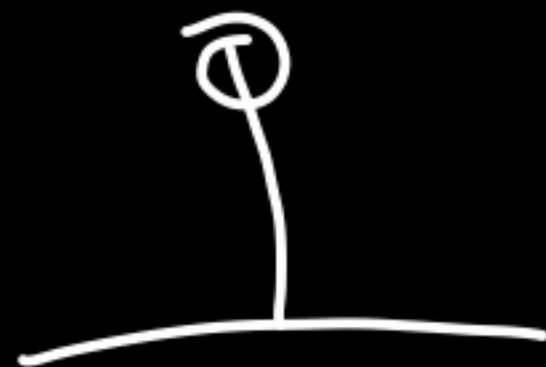
$$x_{k+1} = f(x_k, 0)$$

$$x^{eq} = 0$$

• STABLE



• UNSTABLE



MPC

SYSTEM CONTROLLED WITH MPC

$$x_{k+1} = f(x_k, \underbrace{K_N(x_k)}_{u_{0|k}^*}) = \tilde{f}(x_k)$$

x^{eq} STABLE?

$\sim$
 f x^{eq} COULD CHANGE

QUESTION 1.

IS $x^{eq} = 0$ STILL AN EQUILIBRIUM POINT FOR
THE SYSTEM CONTROLLED WITH MPC?

$$x_{k+1} = f(x_k, u_{0|k}^*) = f(x_k, K_N(x_k)) = \tilde{f}(x_k)$$

GIVEN THE ASSUMPTIONS, IS $x^{eq} = 0$ STILL AN
EQ. POINT FOR $\tilde{f}$

