

14/10/2026 - STABILITY OF MPC.

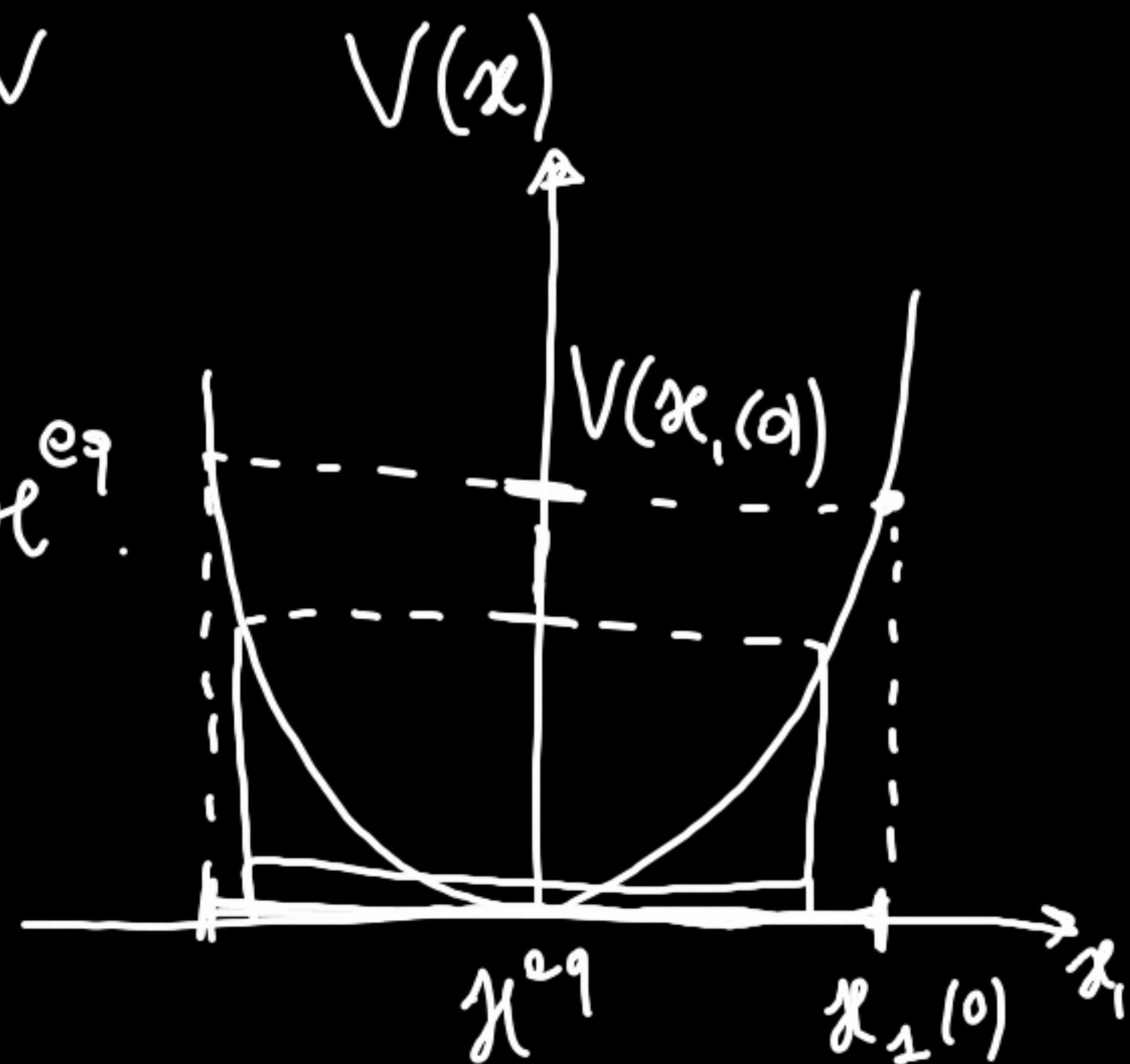
LYAPUNOV TH. ON SIMPLE STABILITY OF x^{eq}

AN EQ. POINT x^{eq} OF $x_{k+1} = f(x_k, u_k)$ IS
STABLE IF WE CAN FIND A FUNCTION

$$V: \mathbb{R}^n \rightarrow \mathbb{R};$$

① $V(x) > 0$ AT LEAST LOCALLY AROUND x^{eq} .

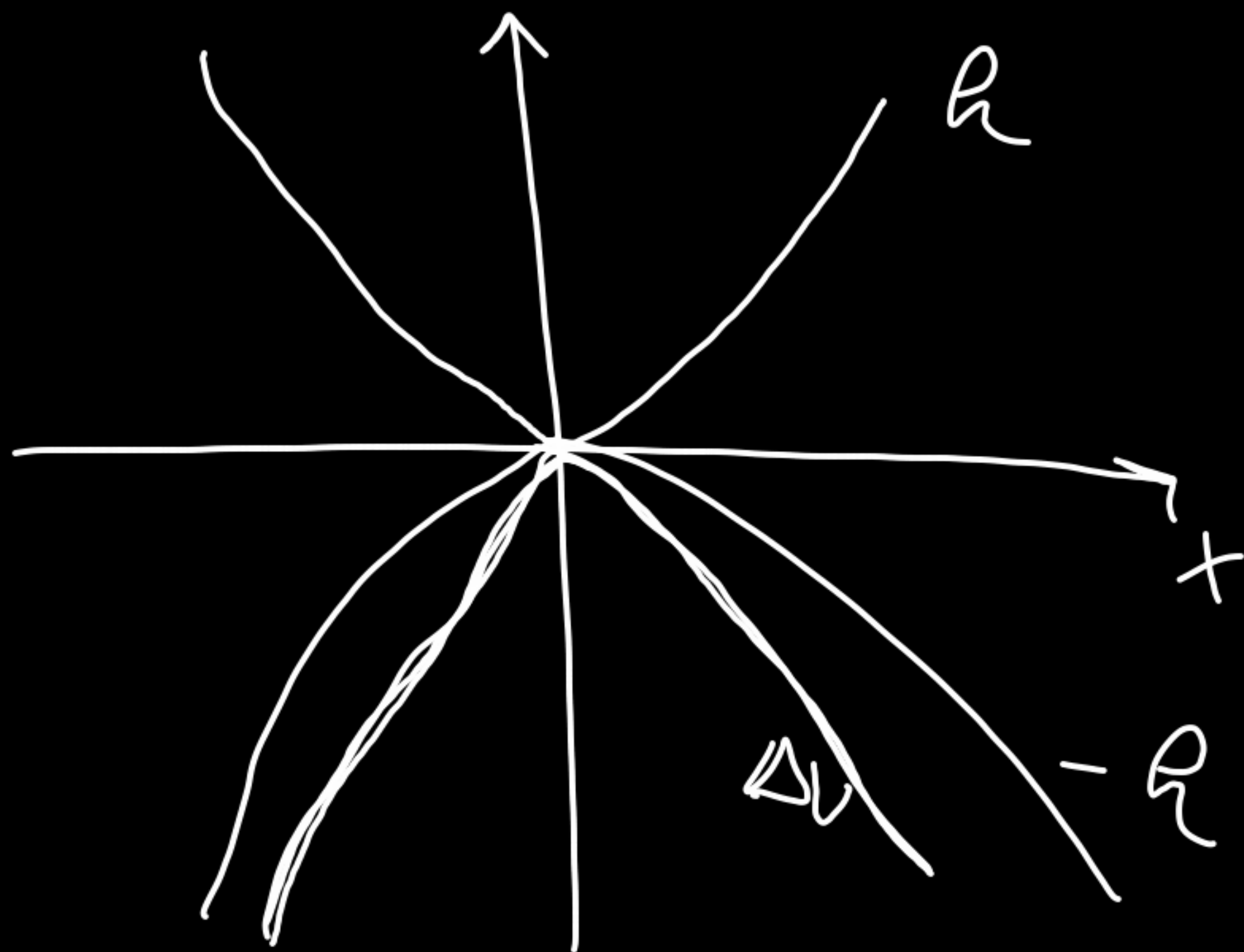
② $\Delta V(x) \leq 0$ / , / / .



$$\textcircled{2} \Delta V \leq 0$$



$$\underline{\Delta V(x) \leq -h(x)}, \text{ WITH } h \geq 0.$$



$$P_N(x_k) : \min_{u_k} \left\{ V_N(x_k, u_k) = \sum_{i=0}^{N-1} \underbrace{\ell(x_{i|k}, u_{i|k})} + \underbrace{V_f(x_{N|k})} \right\}$$

$$u_k \triangleq \{ \underbrace{u_{0|k}}, u_{1|k}, \dots, u_{N-1|k} \}$$

ASSUMPTIONS

$$f(0,0) = 0$$

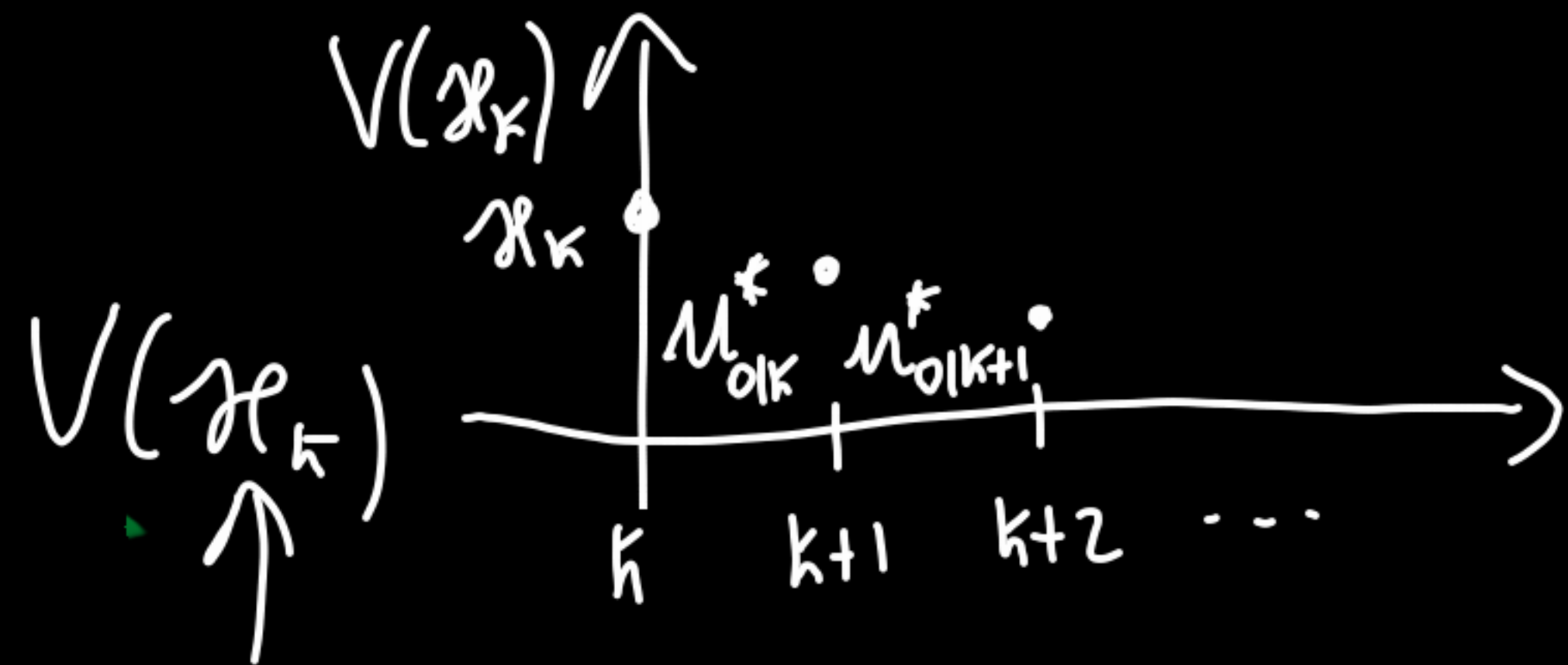
$$\ell(0,0) > 0$$

$$V_f > 0$$

$$s.t. \begin{cases} x_{i+1|k} = f(x_{i|k}, u_{i|k}) ; i=0,1,\dots,N-1 \\ x_{0|k} = x_k \end{cases}$$

$$\begin{cases} \underline{u_{i|k} \in U} ; i=0,1,\dots,N-1 \\ \underline{x_{i|k} \in X} ; i=0,1,\dots,N \end{cases}$$

$$x_{N|k} \in \underline{X_f}$$



CLOSED LOOP SYSTEM AND EQUILIBRIUM



EQ. OF OVERALL CONTROL SYSTEM

$$x_{k+1} = f(x_k, u_{0k}^*) =$$

$$= f(x_k, k_N(x_k)) = \tilde{f}(x_k)$$

$x=0$ IS EQUILIBRIUM OF f . IS IT STILL AN EQ. OF $\tilde{f}$ (f CONTROLLED WITH MPC)?

$$f(0, \underline{k_N(0)}) = 0 \quad ? \checkmark$$

$$k_N(0) = 0$$

|||

$$u_{0k}^*$$

NOW WE SEE A POSSIBLE WAY TO PROVE STABILITY OF $x^e = 0$ FOR A SYSTEM CONTROLLED WITH MPC, USING LYAPUNOV.

WE NEED TO FIND A CANDIDATE LYAPUNOV FUNCTION AND CHECK ① AND ②.

$$V_N(x_k, u_k) \Big|_{u_k = u_k^*} = V(x_k, \{u_{0|k}^*, u_{1|k}^*, \dots, u_{N-1|k}^*\}) = V_N^*(x_k)$$

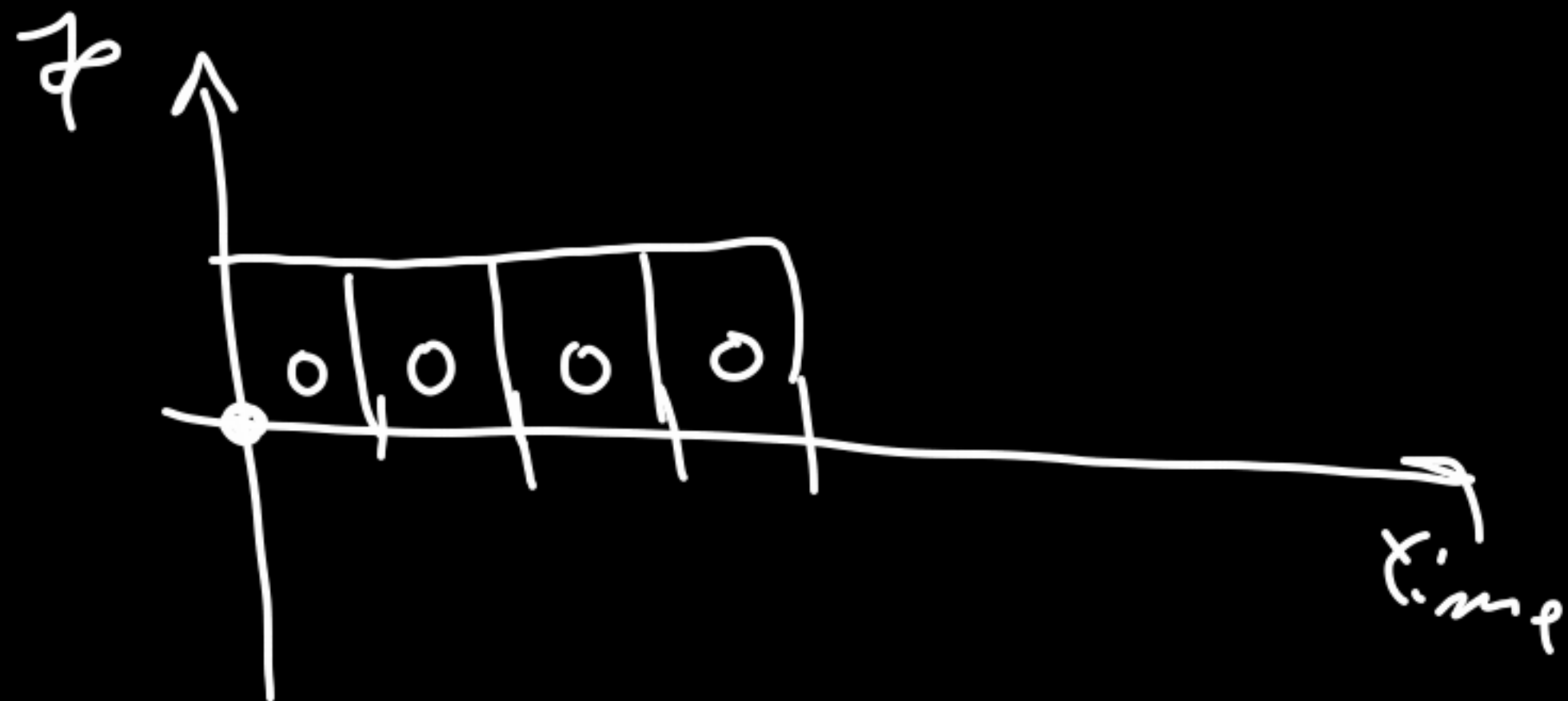
OPTIMAL COST OF MPC ITERATION
STARTING FROM GENERIC x_k .

$$V^*(x_k) = V(x_k, u_k^*)$$

① $V^*(x_k) > 0$? IS V^* A POSITIVE-DEFINITE FUNCTION?

a) $V^*(0) = 0$ ✓ $\ell > 0$

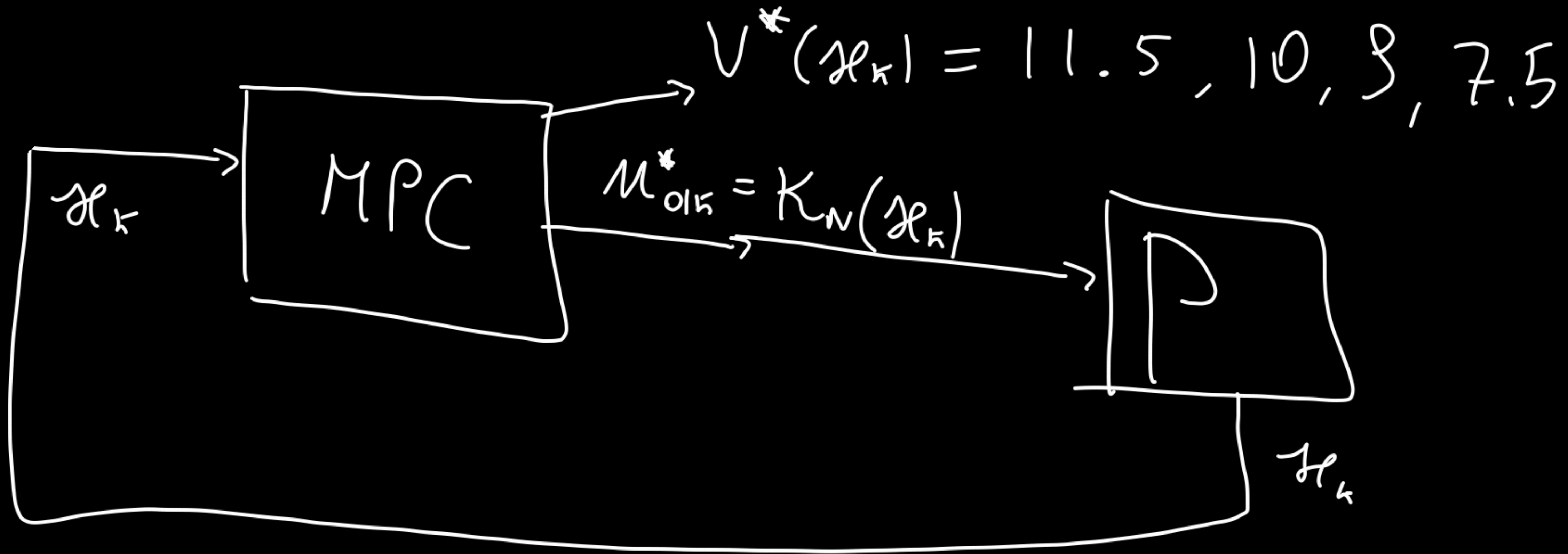
b) $V^*(x) > 0, \forall x \neq 0$ ✓ IF $x_k \neq 0 \rightarrow \ell(x_k, *) > 0$



IF $x_k = 0 \rightarrow \underline{u_k^* = 0, A < 50}$

$x_{0/k}^* = 0, x_{1/k}^* = 0, \dots, x_{N/k}^* = 0$

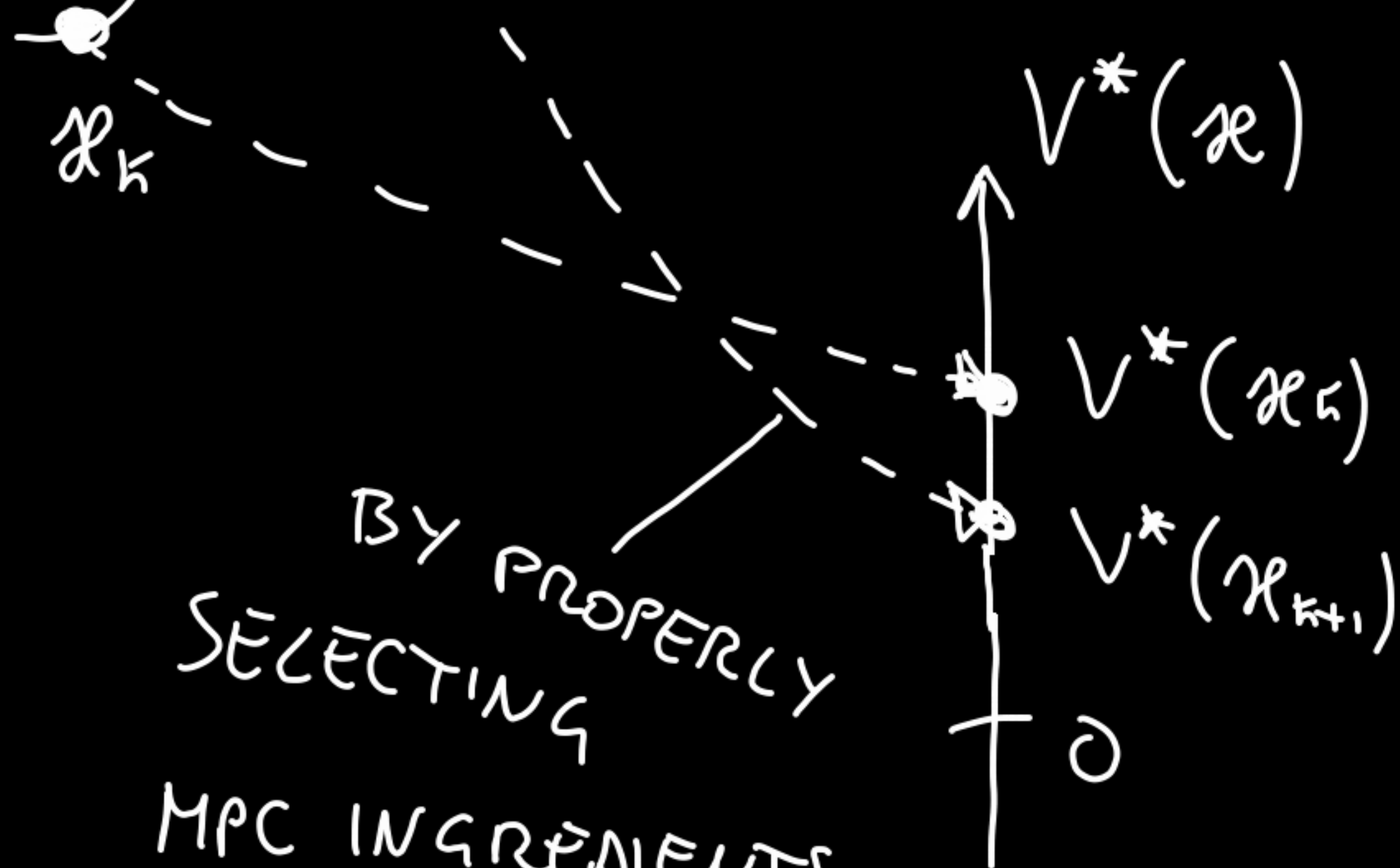
② $\Delta V^*(x_k) \leq 0$? IS THE INCREMENT OF V^* A SEMI-NEGATIVE DEFINITE FUNCTION?



$$\Delta V^*(x_k) = V^*(f(x_k, u_{olk}^*)) - V^*(x_k) \leq 0$$

$$u_{olk}^* = K_N(x_k)$$

$$x_{k+1} = f(x_k, K_N(x_k))$$



BY PROPERLY
SELECTING

MPC INGREDIENTS

BY PROPER
MPC TUNING

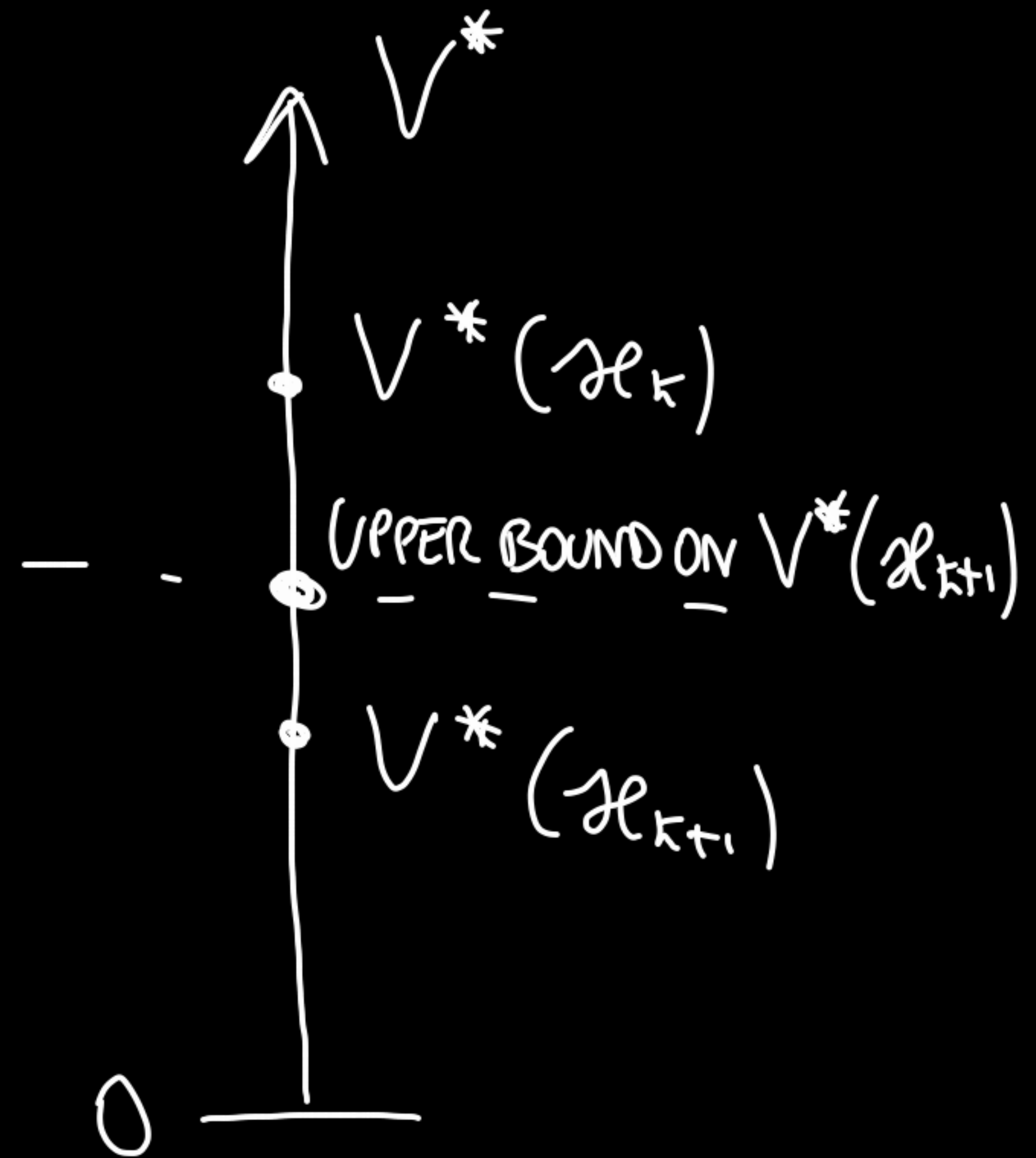
(V_f IN
PARTICULAR)

$$\Delta V^*(x_k) = V^*\left(\underbrace{f(x_k, u_{0|k}^*)}_{x_{k+1}}\right) - V^*(x_k) \leq 0?$$

SOLUTION AT k

$$\{u_{0|k}^*, u_{1|k}^*, \dots, u_{N-1|k}^*\}$$

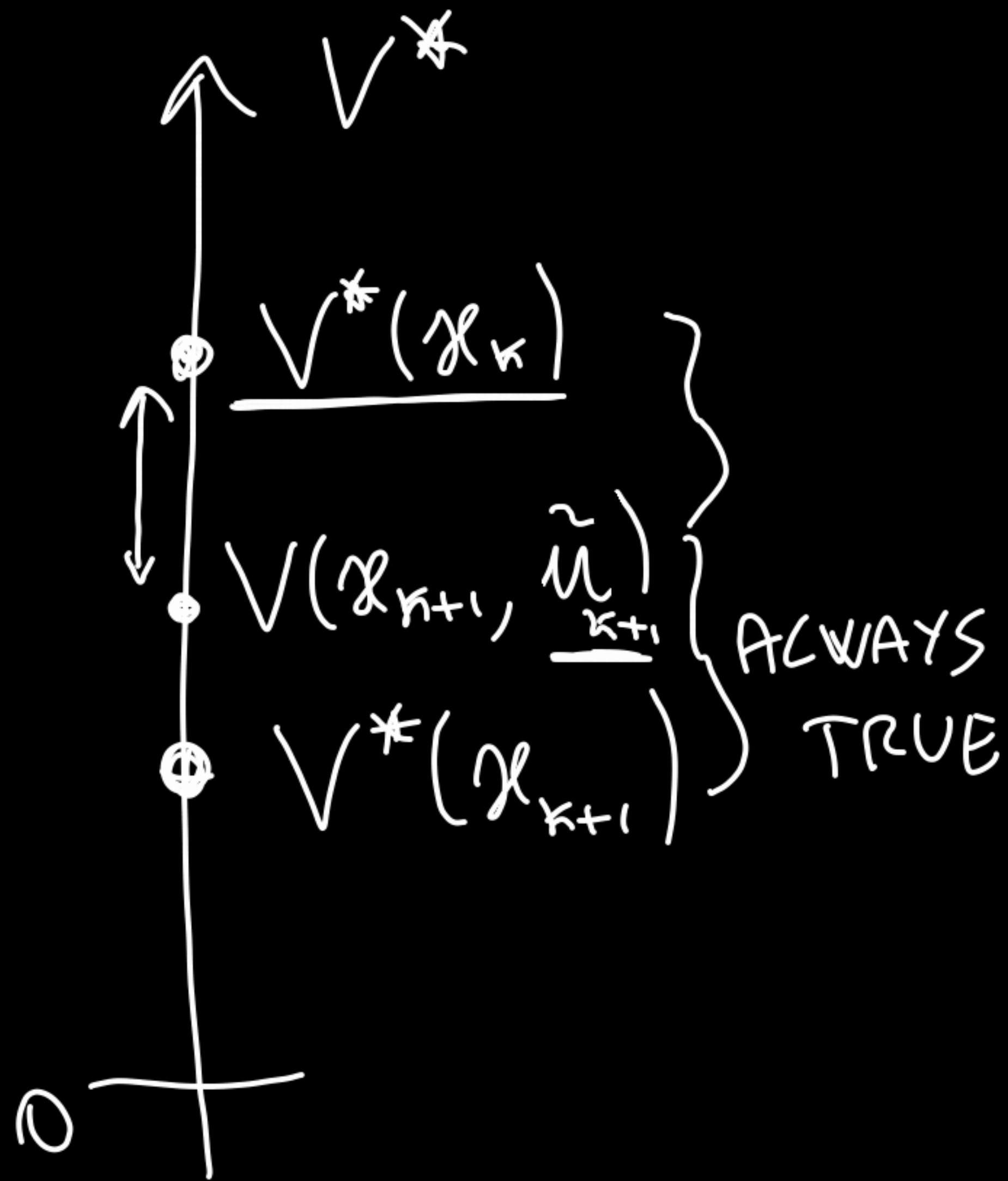
$$\{x_{0|k}^*, x_{1|k}^*, \dots, x_{N|k}^*\}$$



$$\underline{V^*(x_{k+1}) = V^*\left(\underbrace{f(x_k, u_{0|k}^*)}_{x_{k+1}}\right) = V\left(f(x_k, u_{0|k}^*), u_{k+1}^*\right)}$$

$$\leq V\left(f(x_k, u_{0|k}^*), \underline{\tilde{u}_{k+1}}\right), \text{ WITH}$$

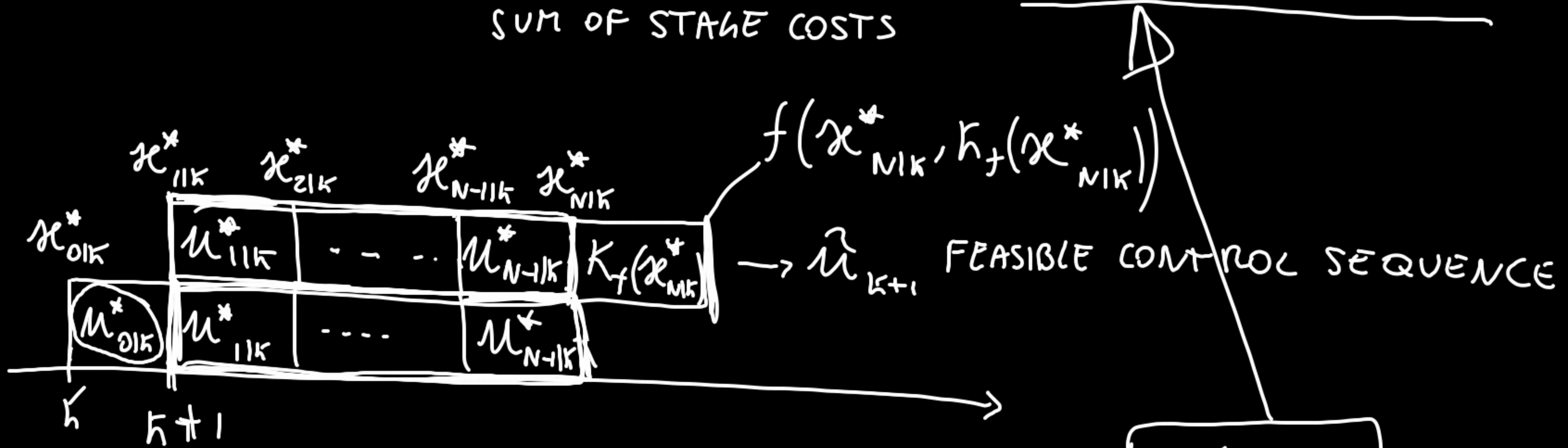
$\tilde{u}_{k+1}$ ANY FEASIBLE SOLUTION AT $k+1$.



NEXT, WE EXPAND $V^*(x_k)$
AND LOOK FOR IDEAS ON
HOW TO SELECT $\tilde{u}_{k+1}$.

NOW CHOOSE $\tilde{u}$ TO KEEP $\rightarrow \Delta V^* \leq 0$.
 $V(x_{k+1}, \tilde{u})$ BELOW $V^*(x_k)$

$$\underline{V^*(x_k)} = \underbrace{\ell(x_{0k}^*, u_{0k}^*) + \ell(x_{1k}^*, u_{1k}^*) + \dots + \ell(x_{N-1k}^*, u_{N-1k}^*)}_{\text{SUM OF STAGE COSTS}} + V_f(x_{Nk}^*)$$



$$\tilde{u}_{k+1} : V(x_{k+1}, \tilde{u}_{k+1}) - V^*(x_k) \leq 0$$

$\tilde{u}_{k+1}$ should make this difference simple

TAIL-BASED $\tilde{u}_{k+1} = \{u_{1k}^*, u_{2k}^*, \dots, u_{N-1k}^*, K_f(x_{Nk}^*)\}$
 $k+1$: LIKE BEFORE

$$V(\underbrace{x_{k+1}}_{x_{k+1} = x_{1k}^*}, \tilde{u}_{k+1}) = \underbrace{\rho(x_{1k}^*, u_{1k}^*) + \rho(x_{2k}^*, u_{2k}^*) + \dots + \rho(x_{N-1k}^*, u_{N-1k}^*)}_{\text{STAGE COSTS}} +$$

(NOMINAL ASSUMPTION)

$$\underbrace{+ \rho(x_{Nk}^*, K_f(x_{Nk}^*))}_{\text{NEW}} + V_f(f(x_{Nk}^*, K_f(x_{Nk}^*)))$$

now $\underline{V(x_{k+1}, \hat{u}_{k+1}) - V^*(x_k)} =$

$$= \rho(x_{nik}^*, K_+(x_{nik}^*)) + \underline{V_f}\left(\underline{f(x_{nik}^*, K_+(x_{nik}^*))}\right) +$$

$$\leq 0$$

$$- \underline{\rho(x_{0ik}^*, u_{0ik}^*)} - V_f(x_{nik}^*)$$

CAN I MAKE THIS ≤ 0 BY PROPERLY

SELECTING V_f AND K_+ ?

$$V(x_{k+1}, \hat{u}_{k+1}) - V^*(x_k) \leq \underbrace{V_f(f(x_{N|k}^*, K_f(x_{N|k}^*))) - V_f(x_{N|k}^*)}_{\text{TERMINAL STATE}} +$$

$$+ \underbrace{\rho(x_{N|k}^*, K_f(x_{N|k}^*))}_{\text{TERMINAL CONTROL}} \leq 0 \quad ?$$

②

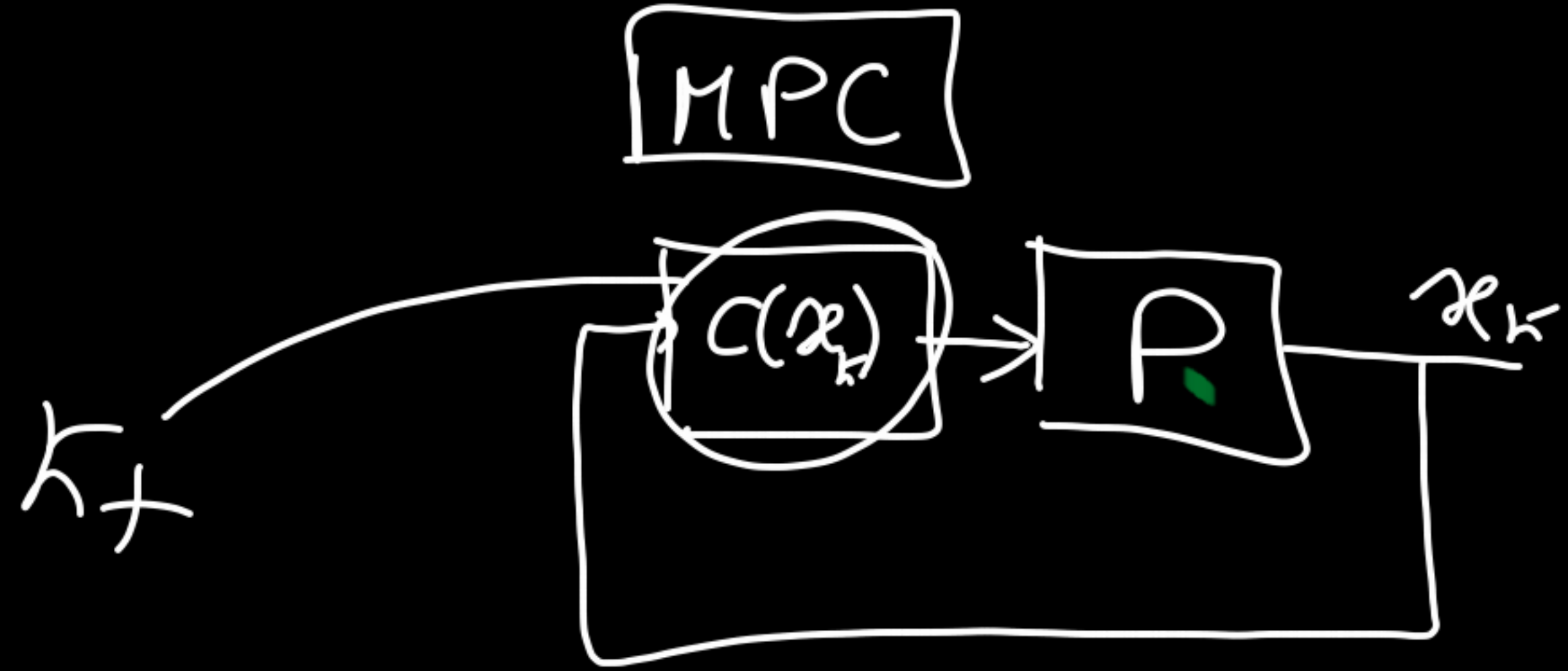
$$\boxed{V_f(f(x_{N|k}^*, K_f(x_{N|k}^*))) - V_f(x_{N|k}^*) \leq -\rho(x_{N|k}^*, K_f(x_{N|k}^*))}$$

IS THIS TRUE? IN WHICH CASES?

$$x_{k+1} = f(x_k, u_k)$$

$$0 = f(0, 0)$$

$$x_{k+1} = f(x_k, c(x_k))$$



FIND A $C(x_k)$ SUCH THAT $x=0$ IS A STABLE EQ.

TO PROVE STABILITY, FIND V :

$$(1) \quad V(x_k) > 0.$$

$$(2) \quad \Delta V(x_k) = V(\overbrace{f(x_k, c(x_k))}^{x_{k+1}}) - V(x_k) \leq -\rho(x_k, c(x_k))$$

